Mathematics 222B Lecture 11 Notes

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February 22, 2022

1 L^2-Based Interior and Boundary Regularity

1.1 H^k elliptic interior regularity

Last time, we were studying L^2-based regularity theory. We were considered with the second order, scalar partial differential operator

$$Pu = -\partial_j(a^{j,k}\partial_k u) + b^j \partial_j u + cu,$$

where $a^{j,k}(x) \succ \lambda I$ for all $x \in U$ for some $\lambda > 0$.

Theorem 1.1 (H^2 interior regularity)**.** *Let U be an open subset of $\mathbb{R}^d$, and suppose $|Da| + |a| + |b| + |c| \leq \Lambda$ for all $x \in U$. Let $u \in H^1(U)$ be a weak solution to $Pu = f$ in U, where $f \in L^2(U)$. Then for all $V \subseteq\subseteq U$ (V bounded with $\overline{V} \subseteq U$), $u \in H^2(V)$, and*

$$\|u\|_{H^2(V)} \leq C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}).$$

Remark 1.1. The constant C is independent of u and f but dependent on λ, Λ, V, U.

The basic ideas in the proof were:

1. Integration by parts and ellipticity give us control over the highest order term.
2. Commute the equation with ∂_j.

In the proof, we looked at the equation for $\partial_j u$, then applied ellipticity to control $\|\zeta D \partial_j u\|_{L^2}$, where ζ was a smooth curoff which equals 1 on B but is 0 near ∂U. In reality, however, to deduce that $u \in H^2(V)$, we have to work with the difference quotient $D_j(u) = \frac{u(x+he_j)-u(x)}{h}$.

Here is the higher regularity version of this theorem.

Theorem 1.2 (H^k elliptic interior regularity)**.** *Assume the same hypotheses as before, except*

- $|D^\alpha a| \leq A$ *for all* $|\alpha| \leq k-1$, $|D^\alpha b| + |D^\alpha c| \leq A$ *for all* $|\alpha| \leq k-2$,
- $f \in H^{k-2}(U)$.

Then for all $V \subseteq\subseteq U$, $u \in H^k(V)$, *and*

$$\|u\|_{H^k(V)} \leq C(\|f\|_{H^{k-2}(U)} + \|u\|_{L^2(U)}).$$

Proof. Here is a sketch. The proof follows the same idea, except we commute D^β for $|\beta| \leq k-1$. Then look at the equation for $D^\beta u$:

$$\begin{aligned} D^\beta f &= D^\beta(Pu) \\ &= D^\beta(-\partial_j(a^{j,k}\partial_k u) + b^j \partial_j u + cu) \\ &= -\partial_j(a^{j,k}\partial_k D^\beta u) + D^\beta(b^j \partial_j u) + D^\beta(cu). \end{aligned}$$

Multiply both sides by $\zeta^2 D^\beta u$. The first term on the right is

$$-\sum_{\gamma \leq \beta, \gamma \neq \beta} \partial_j(D^{\beta-\gamma} a^{j,k} \partial_k D^\gamma u) c_\gamma.$$

This gives us control of $\|DD^\beta u \zeta\|_{L^2(U)}$. For the rest of the terms, you do not see more than $k-1$ derivatives of of u and $k-2$ derivatives of b and c after integration by parts.

In reality, the details need to be carried out with difference quotients, using induction to take care of lower derivative terms. The full proof is in Evans' book. $\square$

1.2 L^2-based boundary regularity

Previously, we have been looking at regularity away from the boundary. You may also notice that we have not been putting conditions on boundary behavior of u (we only required, for example, $u \in H^1$ rather than $u \in H^1_0$).

Theorem 1.3. *Assume the same hypotheses as in the* H^2 *interior regularity theorem, except:*

- $u \in H^1_0(U)$ *(i.e.* $u|_{\partial U} = 0$ *in the sense of traces).*
- ∂U *is* C^2.

Then $u \in H^2(U)$, *and*

$$\|u\|_{H^2(U)} \leq C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}).$$

Proof. Assume for simplicity that $u \in H^2(U)$; we can take care of this by doing the argument with difference quotients instead of derivatives. We will omit the contribution of b and c because they do not contribute much, as we have seen. Start with the equation

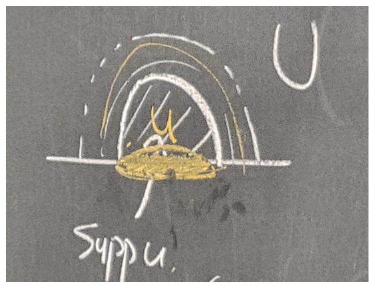
$$f = \partial_j(a^{j,k}\partial_k u) + \cdots .$$

We want to take a derivative to say

$$\partial_\ell f = -\partial_\ell(\partial_j(a^{j,k}\partial_k u)),$$

but we cannot necessarily take the derivative at the boundary. However, notice that if the boundary is flat (wlog $\{x^d = 0\}$), then all ∂_ℓ exist for $\ell = 1, \ldots, d-1$. The only problem is the normal derivative $\partial_{x^d} = -\nu$. In other words only $(d-1)$-many directions (tangential to ∂U) are admissible.

For the sake of simplicity, take the special case when $U = B_1(0) \cap \mathbb{R}^d_+$ and $\operatorname{supp} u \subseteq B_{1/2}(0) \cap \mathbb{R}^d_+$.



In this case, $\ell f = -\partial_j(a^{j,k}\partial_k\partial_\ell u) - \partial_j(\partial_\ell a^{j,k}\partial_k u)$ for $\ell = 1, \ldots, d-1$. For these $d-1$ terms, we can use the cutoff ζ which equals 1 on $B_{1/2}(0)$ and is 0 near $\partial B_1(0)$ to get

$$\|\zeta D\partial_\ell u\|_{L^2} \leq C(\|\zeta f\|_{L^2} + \|u\|_{L^2}).$$

In the integration by parts, there is an additional boundary term from $B_{1/2}(0) \cap \{x^d = 0\}$. However, this contribution is zero because $u|_{\partial U} = 0$, which also implies $\partial_\ell u|_{\partial U} = 0$ for $\ell = 1, \ldots, d-1$.

In this special case, it now remains to control $\|\zeta\partial_{x^d}\partial_{x^d}u\|_{L^2}$. The key observation is that the equation allows us to express $D_\nu D_\nu u$ in terms of everything else. Recall that the original equation is

$$f = -\partial_j(a^{j,k}\partial_k u) + \cdots.$$

The condition that $a \succ \lambda I$ is equivalent to $a^{j,k}\xi_j\xi_k \geq \lambda|\xi|^2$ for all $\xi \in \mathbb{R}^d$. If we take $\xi = e_d$, this tells us that $a^{d,d} \geq \lambda$. Now write the equation as

$$f = \underbrace{-\partial_d(a^{d,d}\partial_d u)}_{=a^{d,d}\partial_d^2 u - (\partial_d a^{d,d})\partial_d u} - \sum_{\substack{j,k \\ j,k \neq d}} \partial_j(a^{j,k}\partial_k u).$$

We can divide the equation by $a^{d,d}$ to get

$$\partial_d^2 u = \frac{1}{-a_{d,d}}(a \cdot D_{\tan} u + (\partial a), b)Du + cu + f.$$

This lets us control $\partial_d^2 u$ by the other derivatives, completing the proof in this special case.

In general, we reduce to this special case by first using a smooth partition of unity and boundary straightening. In particular, for every $x \in \partial U$, there exists a ball $B_r(x)$ such that, after relabeling of the coordinate axes, $U \cap B_r(x) = \{x^d > \gamma(x^1, \ldots, x^{d-1})\}$ for some C^2 function γ. We then take a boundary straightening map y, defined by

$$\begin{cases} y^\ell = x^\ell & \ell = 1, \ldots, d-1 \\ y^d = x^d - \gamma(x^1, \ldots, x^d). \end{cases}$$

By compactness, $U \subseteq (\bigcup_{k=1}^K U_k) \cup U_0$, where U_k are balls covering the boundary and U_0 contains the rest of the interior. Then there exists a smooth partition of unity $\{\chi_k\}_{k=0}^K$ subordinate to this cover, which gives

$$u = \chi_0 u + \sum_{k=1}^K \chi_k u.$$

The first term is supported on the interior of U, so we can apply our interior regularity theorem to it. For each other $\chi_k u$, when we change $x \mapsto y = y(x)$, we are reduced to the half-ball case already covered (both in terms of geometry and support of u). Check that the ellipticity constant of the resulting equation is still $\simeq \lambda$ and that $\partial \widetilde{a}(y)$, $\widetilde{b}(y)$ obey same bounds as before; this comes from writing the equation in terms of derivatives in y and checking that the change of variables formula $a^{j,k} = \frac{\partial x^j}{\partial y^{j'}} \widetilde{a}^{j',k'} \frac{\partial x^k}{\partial y^{k'}}$ preserves the $a \succ \lambda I$ condition. From the H^2 bound for $u\chi_k(y)$, come back to $u\chi_k(x)$ (which needs the C^2 condition on ∂U). $\square$

1.3 High level comparison of L^2-based regularity theory and Schauder theory

L^2-based regularity theory, which deals with weak solutions in H^1, is useful for deriving the existence of the solution. In order to derive the H^1 bound, we only need $a \in L^\infty$, rather than requiring additional regularity. Think of

$$-\partial_j(a^{j,k}(u)\partial_k u) = f,$$

where the coefficients $a^{j,k}$ may be very rough. However, it is wasteful in terms of the regularity required of a for higher regularity of the solution u.

To rectify this, we want another regularity theory that works well in this respect for nonlinear equations. This is achieved by Schauder theory, elliptic regularity theory in $C^{k,\alpha}$. Hölder spaces are naturally algebras; they play well with products, which are generally the problem with nonlinear PDEs. The gap between L^2-based regularity theory and Schauder theory is given by the famous de Giorgi-Nash-Moser estimates, which we will hopefully discuss later in the course.